ALCUNI CONTRIBUTI DI GIOVANNI CELENTANO E ALDO BALESTRINO

SULLA SINTESI DIRETTA DEI SISTEMI MULTIVARIABILI

Prof. G. Celentano

Breve storia

Incontrai il Prof. Aldo Balestrino al III anno dei miei studi di ingegneria in occasione di un'esercitazione di Metodi Matematici nel 1971. Ricordo che sviluppò un esercizio in tre modi diversi, anche se molti studenti non se ne resero conto. Poi lo rincontrai al V anno nel 1973 in qualità di professore di Complementi di Controllo. Al termine dell'esame in questione, il Prof. Lorenzo Sciavicco mi chiese se volevo fare la tesi con lui o con il Prof. Balestrino per poi, eventualmente, entrare a far parte del loro gruppo di ricerca. Anche se mi era stata fatta una proposta analoga dal Prof. F. Gasparini per l'Università di Cosenza, accettai per la grande fiducia che avevo in loro e per la loro onestà intellettuale. Dal 1974 incominciai a lavorare soprattutto con il Prof. A. Balestrino, mentre, successivamente, Giuseppe De Maria prevalentemente con il Prof. L. Sciavicco e Pompeo Marino prevalentemente con il Prof. C. Vicinanza.
Ci ponemmo come obiettivo quello di far diventare la sede napoletana una scuola di serie A nel campo dell'Automatica sia per i ricercatori che per gli studenti.
Il primo anno fu soprattutto di studio e di esplorazione di temi della moderna teoria del controllo (analisi funzionale, stabilità dei sistemi lineari e non, sintesi dei sistemi multivariabili nel dominio del tempo). Ebbi la conferma che Balestrino era uno studioso che conosceva la matematica e per la quale nutriva grande passione. Notai anche che era molto curioso e voleva che ci si occupasse di molti argomenti.
In meno di un anno raggiunsi un livello di preparazione e di conoscenza della moderna teoria del controllo tale da poter iniziare a competere con gli altri ricercatori a livello internazionale. Infatti i primi risultati non tardarono ad arrivare.
Ricordo il momento in cui il Prof. Guido Guardabassi ci informò che il lavoro "Stabilization by digital controllers of multivariable linear systems with time-lags" era stato accettato per il 7th IFAC Triennal World Congress.

Balestrino, contentissimo, mi disse: “ora siamo alla pari con gli altri più accreditati ricercatori mondiali”.
Sul fronte della didattica scrivemmo dei buoni testi e numerose dispense. Incominciammo ad essere un punto di riferimento di moltissimi studenti validi. Tutti volevano fare la tesi con il nostro gruppo. Ci fu un anno in cui seguimmo circa cinquanta tesisti (il Prof. Sciavicco fu costretto a tenere un incontro con i professori degli altri gruppi disciplinari per limitare l’afflusso di tesisti verso il nostro gruppo)! Ciò consentì a Sciavicco e a Balestrino di reclutare altri giovani validissimi: B. Siciliano, P. Chiacchio, S. Chiaverini, L. Glielmo, F. Amato, A. Pironti, L. Villani, … .
Dopo che Sciavicco e Balestrino diventarono professori ordinari le nostre strade incominciarono a dividersi. Io prosegui a fare ricerca con Ambrosino e Garofalo (che erano tornati da Milano), con Amato, Cavallo, Setola, Mattei ed altri, mentre Balestrino continuò con Sciavicco, De Maria e numerosissimi altri.
Fummo sempre veri amici, amicizia che si estese anche alle nostre famiglie.

In seguito si riporta l’elenco delle pubblicazioni fatte con Aldo Balestrino.

Pubblicazioni scientifiche di rilevante interesse

1. A. Balestrino, G. Celentano, L. Sciavicco, “On incomplete pole assignment in linear systems”, *Systems Science III*, International Conference on Systems Science, Wroclaw, 1976.

2. A. Balestrino, G. Celentano, L. Sciavicco, “On incomplete pole assignment in linear systems”, *Systems Science*, vol. 2, n. 1, 1976.

3. A. Balestrino, G. Celentano, “On the structural properties and on the input and output reducibility of multivariable linear systems”, *Ricerche di Automatica*, vol. n. 7, n. 2-3, 1976.

4. A. Balestrino, G. Celentano, L. Sciavicco, “Asymptotic stability regions for classes of nonlinearities”, *Ricerche di Automatica*, vol. n. 8, n. 2-3, 1977.

5. A. Balestrino, G. Celentano, “Pole assignment in linear multivariable systems using compensator of reduced order”, *Ricerche di Automatica*, vol.8. n. 2-3, 1977.

6. A. Balestrino, G. Celentano, “Stabilization by digital controllers of multivariable linear systems with time-lags”, *7th Triennal World Congress,* IFAC-Helsinki, vol. 3, Pergamon Press, Oxford, 1978.

7. A. Balestrino, G. Celentano, “Pole assignment in linear multivariable systems using observer of reduced order”, *IEEE Trans. on Automatic Control*, vol. AC-24, n. 1, 1979.

8. A. Balestrino, G. Celentano, “C.A.D. of minimal order controllers”, *Computer Aided Design of Control Systems IFAC Symposium,* Zurich, Pergamon Press, Oxford, 1979.

9. A. Balestrino, G. Celentano, “Design of PD controllers in linear multivariable systems”, *Proc. IEE*, vol. 127, part D, n. 3, 1980.

10 A. Balestrino, G. Celentano, “Comments on Pole assignment and determination of the residual polynomial”, *IEEE Trans. on Automatic Control*, vol. AC-25, n. 1, 1980.

11 A. Balestrino, G. Celentano, “Dynamic controllers in linear multivariable systems”, *Automatica*, vol.17, n. 4, 1981.

12. A. Balestrino, G. Celentano, “Design of PID controllers in linear multivariable systems”, *Ricerche di Automatica*, vol. 14, n. 1, 1983.

13. G. Celentano, A. Balestrino, “New techniques for the design of observers”, *IEEE Trans. on Automatic Control*, vol. AC-29, n. 9, 1984.

Rapporti interni di tipo scientifico

1. A. Balestrino, G. Celentano, “An algorithm to compute the minimal polynomial of a matrix”, 1981.

2 . A. Balestrino, G. Celentano, “Properties of matrix equations in root clustering and related problems”, 1984.

Libri didattici

1. A. Balestrino, G. Celentano, "Teoria dei Sistemi : Definizioni e proprietà dei sistemi dinamici", vol. I, *Liguori Editore*, Napoli, 1979.

2. A. Balestrino, G. Celentano, "Teoria dei Sistemi: I sistemi a stati finiti", vol. II, *Liguori Editore*, Napoli, 1981.

3. A. Balestrino, G. Celentano, "Teoria dei Sistemi: I sistemi dinamici a stato vettore", vol. III, *Liguori Editore*, Napoli, 1982.

Rapporti interni di tipo didattico-divulgativo

1. A. Balestrino, G. Celentano, "Proprietà strutturali dei sistemi dinamici", 1979.

2. A. Balestrino, G. Celentano, "Elementi di stabilità dei sistemi dinamici", 1979.

3. A. Balestrino, G. Celentano, "Equazioni generalizzate di Lyapunov", 1981.

In seguito si riportano i **primi due lavori più significativi** pubblicati con Aldo Balestrino.

Numerosi risultati dell'attività di ricerca con Balestrino si possono trovare nel mio libro "**Elementi di sintesi diretta dei sistemi multivariabili**", *Liguori Editore, 1981*.
In seguito si riporta anche una **parte del capitolo VI** di tale libro dal titolo "Controllori dinamici interagenti".

ON INCOMPLETE POLE ASSIGNMENT IN LINEAR SYSTEMS

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Abstract. New theorems and new algorithms are reported in this paper for pole assignment in a dynamic system through output feedback. Such results allow the assignment of some poles with the other closed loop poles simultaneously singled out: moreover, degrees of freedom in synthesis are pointed out, whenever they turn out to be useful for setting adequate limits to non preassigned poles.

1 Introduction

Consider a linear and time invariant dynamic system, described by the triplet (*C*, *A*, *B*), where *A* is the dynamic $n \times n$ matrix of the system, *B* the $n \times p$ input matrix, and *C* the $m \times n$ output matrix. It is well known that if the system is reachable, it is possible to assign an arbitrary symmetric spectrum to the closed loop dynamic matrix by selecting an appropriate state feedback [1]; if the state is not accessible, use may be made of a dynamic observer [2] or of a dynamic compensator [3]. From a practical point of view the problem of assigning closed loop eigenvalues through output feedback is relevant. Latest results concerning this problem, which has been faced by many authors, are found in [4] and [5]; it was proved therein that if the triplet (*C*, *A*, *B*) is complete, with $\text{rank} B = p \le n$, with $\text{rank} C = m \le n$, then by using output feedback min(*n*, *q*) eigenvalues, $(q = m + p - 1)$ for almost all (*B*, *C*) pairs, can be assigned arbitrarily close to min(*n*, *q*) specified symmetric values. These results are not practical if $q < n$; in this case, in fact, nothing is said about the remaining eigenvalues or the region of the complex plane they are confined to.

In this paper a new procedure is presented allowing an alternative proof of DAVISON'S theorem [5]. The algorithm developed is more effective than previous ones, and gives the polynomial whose roots are the remaining closed loop eigenvalues, if $q<n$. Moreover, such an algorithm is extended to the problem of an approximate setting of more than q eigenvalues as well as to the exact assignment of less than q eigenvalues, if $q<n$. In the last case degrees of freedom in synthesis are pointed out whenever they turn out to be useful for setting adequate limits to non preassigned eigenvalues.

2 Problem statement and preliminary results

Let a linear and time invariant dynamic system, completely reachable and observable, be described by:

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{1}$$

where *A*, *B*, *C* are $n \times n$, $n \times p$, and $m \times n$ constant real matrices, respectively; moreover, rank$B = p$, rank$C = m$.
If an output feedback

$$u = Kx \tag{2}$$

is applied to system (1) the dynamic matrix $\hat{A}$ of the closed loop system becomes

$$\hat{A} = A + BKC. \tag{3}$$

Let us first prove the following result [6], [7]:

Theorem 1. Let system (1) be given. An output feedback gain matrix $K \in R^{p \times n}$ can always be found so that matrix (3) has max(m,p) eigenvalues assigned arbitrarily close to max(m,p) specified symmetric values.

Proof. Let us consider only the case where $m \geq p$, since for $m < p$, the same procedure could be reiterated by considering, instead of (3), the corresponding transposed [6]. Let us also assume that $B = b$, where *b* is an *n*-vector; otherwise it should be simply recalled that for almost any matrix *K* and any vector $w \in R^p$ the pair $(A + BKC, Bw)$ is reachable [7], [8].
Let us denote by

$$p_{\mathbf{A}}(s) = s^n + a_1 s^{n-1} + \ldots + a_n \tag{4}$$

the characteristic polynomial of matrix *A*, with

$$\hat{p}_{\mathbf{A}}(s) = s^n + \hat{a}_1 s^{n-1} + \ldots + \hat{a}_n \tag{5}$$

the characteristic polynomial of matrix $\hat{A}$; it is well known that [9] the two polynomials (4) and (5) are related by

$$\hat{p}_{\mathbf{A}}(s) = p_{\mathbf{A}}(s) - p_{\mathbf{A}}(s) k^T C (sI - A)^{-1} b \,. \tag{6}$$

Eq. (6), by means of SOURIAU formula [10], can be rewritten as

$$\hat{p}_{\mathbf{A}}(s) = p_{\mathbf{A}}(s) - k^T C \sum_{i=0}^{n-1} s^{n-1-i} N_i b \,, \tag{7}$$

where

$$N_i = A^i + a_1 A^{i-1} + \ldots + a_i I \,. \tag{8}$$

By equating polynomials at left and right hands of (7), taking into account (8), we have

$$Fk = a - \hat{a} \tag{9}$$

where *F* is a $n \times m$ matrix given by

$$F = \begin{bmatrix} a_{n-1} & a_{n-2} & . & a_1 & 1 \\ a_{n-2} & a_{n-3} & . & 1 & 1 \\ . & . & . & . & . \\ a_1 & 1 & . & 0 & 0 \\ 1 & 0 & . & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} Cb & CAb & \ldots & CA^{n-1}b \end{bmatrix}^T \tag{10}$$

and:

$$a = \begin{bmatrix} a_n & a_{n-1} & \dots & a_1 \end{bmatrix}^T \tag{11}$$

$$\hat{a} = \begin{bmatrix} \hat{a}_n & \hat{a}_{n-1} & \dots & \hat{a}_1 \end{bmatrix}^T . \tag{12}$$

Eq. (9) shows that the vector $\hat{a}$, which specifies the characteristic polynomial of $\hat{A}$ by varying *k*, describes a linear variety coinciding with R^n if and only if the pair (*A*, *b*) is reachable and $\text{rank}C = n$. If $\text{rank}\,C < n$, not all closed loop eigenvalues can be arbitrarily assigned.
First of all let us prove that *m* eigenvalues arbitrarily close to *m* pre-assigned symmetric values can be assigned.
Let

$$p_{\mathbf{d}}(s) = s^m + d_1 s^{m-1} + \ldots + d_m \tag{13}$$

be the polynomial having as its roots the desired eigenvalues and

$$p_{\mathbf{r}}(s) = s^{n-m} + r_1 s^{n-m-1} + \ldots + r_{n-m} \tag{14}$$

be the polynomial which, multiplied by $p_{\mathbf{d}}(s)$ results in $\hat{p}_{\mathbf{A}}(s)$.
The following relation

$$\hat{a} = Gr + d \tag{15}$$

can be easily checked, where G is an $n \times (n-m)$ matrix given by

$$G = \begin{bmatrix} 0 & . & . & d_m \\ . & . & . & d_{m-1} \\ d_m & d_{m-1} & . & . \\ . & . & . & . \\ d_1 & 1 & . & 0 \\ 1 & 0 & . & 0 \end{bmatrix} \tag{16}$$

and:

$$r = \begin{bmatrix} r_1 & r_2 & \dots & r_{n-m} \end{bmatrix}^T \tag{17}$$

$$d = \begin{bmatrix} 0 & \dots & 0 & d_m & \dots & d_1 \end{bmatrix}^T .$$
(18)

By replacing (15) in Eq. (9), the following linear relation is obtained

$$\begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} k \\ r \end{bmatrix} = a - d . \tag{19}$$

If matrix $\begin{bmatrix} F & G \end{bmatrix}$ is invertible Eq. (19) allows to compute feedback vector k by means of which the desired closed loop eigenvalues are obtained, as well as the coefficients of polynomial $p_{\mathrm{r}}(s)$.

As far as invertibility of matrix $\begin{bmatrix} F & G \end{bmatrix}$ is concerned, the following theorem holds.

Theorem 2. The determinant of $\begin{bmatrix} F & G \end{bmatrix}$ is a polynomial in variables d_i of degree $n-m$, which does not vanishes for almost any choice of variables d_i, $i = 1, \dots, m$.

Proof. On the basis of Laplace's rule, we have

$$\det \begin{bmatrix} F & G \end{bmatrix} = h_1 g_1 + k_2 g_2 + \dots + h_{\binom{n}{m}} g_{\binom{n}{m}} , \tag{20}$$

where h_i and g_i , $i = 1, 2, \dots, \binom{n}{m}$, are *m* order and $n-m$ order minors of matrices *F* and *G*, respectively. By iterating on $n-m$ and by Laplace's rule there ensures that $\binom{n}{m}$ minors of *G* of highest order are polynomials, with respect to variables d_i, of degree j, $j = 0, 1, ..., n-m$; there are, moreover, $\binom{m+j-1}{j}$ polynomials of degree *j*, and, by neglecting sign, they are given as

$$d_{i_1} \times d_{i_2} \times \ldots \times d_{i_j} + ****, \tag{21}$$

where:

$$\begin{aligned} & i_1 = 1, 2, ..., m \\ & i_2 = i_1, i_1 + 1, ..., m \\ & .... \\ & i_j = i_{j-1}, i_{j-1} + 1, \ldots, m \end{aligned} \tag{22}$$

and stars in (21) denote a non-specified linear combination of previous polynomials. Obviously, these polynomials are linearly independent; on the other hand,(20) assuming that system (1) is reachable and observable, since *F* has maximum rank, at least a minor h_i must be not zero and therefore polynomial (20) cannot be identically null.
From the above theorem it follows that if $\det\begin{bmatrix} F & G \end{bmatrix}$ vanishes for a set of eigenvalues, and therefore of coefficients d_i, a new set of eigenvalues, arbitrarily close to previous ones can be assigned so that $\det\begin{bmatrix} F & G \end{bmatrix} \neq 0$.

3 Some extensions

The case is now considered where ν eigenvalues of $\hat{A}$ should be assigned, with $\nu \neq m$.

Case a.
If $\nu > m$, it is not always possible, by choosing *k*, to obtain the desired eigenvalues; in this case an equation similar to (19) is obtained, with *G* a $n \times (n-\nu)$ matrix, and therefore [*F*, *G*] is a $n \times (n-\nu+m)$ matrix.
If $a - d \in \Re\begin{bmatrix} F & G \end{bmatrix}$, then a feedback vector *k* can be found, so that ν eigenvalues of $\hat{A}$ coincide with ν preassigned symmetric values;

otherwise. Eq. (19) can be approximately solved, for instance by minimizing a suitable norm of the vector difference between left and right hand vectors in (19).
It is useful to note here that

$$\text{probability}\left\{\text{rank}\left[F \quad G\right] = n - \nu + m\right\} = 1 . \tag{23}$$

In fact, by adjoining $\nu - m$ columns to matrix F so that the augmented matrix has ν column-vectors linearly independent, the case where $\nu = m$ is again obtained, and therefore (23) follows.

Case b.
Far more important is the case where $\nu < m$. Here, again, we have an equation of type (19), with G a $n \times (n - \nu)$ matrix and therefore with and therefore $[F, G]$ a $n \times (n - \nu + m)$ matrix. It is observed that:

$$\text{probability}\left\{\text{rank}\left[F \quad G\right] = n\right\} = 1 . \tag{24}$$

In fact, by eliminating $\nu - m$ rows from F, the case comes up again where $\nu = m$ and therefore (24) follows.
Feedback vector k, which allows to attain desired closed loop eigenvalues is not unique, and Eq. (19) can be solved for $k_{m-\nu+1}, \ldots, k_m, r_1, r_2, \ldots, r_{n-\nu}$, which are unknown as functions of the variables: $k_1, k_2, \ldots, k_{m-\nu}$. Then vector r may be written as

$$r = c_0 + k_1 c_1 + \ldots + k_{m-\nu} c_{m-\nu} , \tag{25}$$

where c_i are $(n-\nu)$-vectors and $k_i, i = 1, 2, \ldots, m - \nu$, are feedback coefficients, which can be arbitrarily set without altering prefixed eigenvalues.
Eq. (25) shows that, by adequately choosing $k_i, i = 1, 2, \ldots, m - \nu$, it is possible to determine polynomials (14) having their roots in prefixed regions of the complex plane. For this purpose, the root locus technique can be repeatedly used.
The following example is given to stress the efficiency of the method suggested.

Example 1. Let a reachable and observable system be described by:

$$A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\quad b=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad C=\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}; \tag{26}$$

it is intended to determine an output feedback making the system asymptoticallv stable with an eigenvalue equal -3.
Eq. (19) is specialized into

$$\begin{bmatrix}1&0&0&3\\0&1&3&1\\0&1&1&0\end{bmatrix}\begin{bmatrix}k_1\\k_2\\r_1\\r_2\end{bmatrix}=\begin{bmatrix}0\\0\\-3\end{bmatrix}, \tag{27}$$

which, solved for k_2, r_1, r_2, gives:

$$k_2=-4.5-k_1/6 \tag{28}$$

$$\begin{bmatrix}r_1\\r_2\end{bmatrix}=\begin{bmatrix}1.5\\0.0\end{bmatrix}+\begin{bmatrix}1/6\\-1/3\end{bmatrix}k_1. \tag{29}$$

The closed loop dynamic matrix has one eigenvalue in -3, and the remaining two eigenvalues are roots of the equation

$$s^2-1.5s+k_1(s/6-1/3)=0, \tag{30}$$

whose root locus is given in Fig. 1 for positive and negative k_1. It should be noted that if the method described in [5] had been applied to assign exactly two eigenvalues (one in -3 and the other one less then -1.5), then a real positive number have resulted as the third eigenvalue.

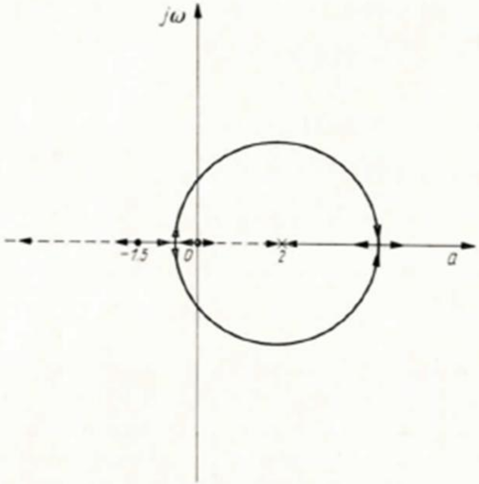

Fig. 1. Root locus relative to example 1
——— $k_1 < 0$
·········· $k_1 > 0$

4 Further results

Theorem 3. Let .system (1) be given; for almost any pair (*B*, *C*) there exists such a matrix *K* that $A+BKC$ has min(n, q) eigenvalues assigned arbitrarily close to min(n, q) specified symmetric values.

Proof. If $\min(m,p)=1$, then $\min(n,q)=\max(m,p)$ and Theorem 3 comes from Theorem 1. Assuming $\min(m,p)>1$, the following preliminary results are established.

Lemma 1. Let a pair (*A*, *B*) be given, *A* and *B* being the respective $n\times n$ and $n\times p$ matrices, with $\operatorname{rank} B = p$. Assuming that *A* has $l=n-p+1$ eigenvalues λ_i, with $i=1,2,...,l$ symmetric with unitary geometric multiplicity, and that the polynomial

$$p_l(s)=s^l+a_1s^{l-1}+\ldots+a_l \tag{31}$$

has such eigenvalues as its roots, then for almost any matrix B there exist a vector $b \in \Re(B)$ so that $p_l(\lambda_i)$ is the minimal polynomial of b with respect to A.
The lemma is proved if exist a non null p-vector w such that:

$$\begin{aligned} &i) \quad p_l(A)Bw = 0, \\ &ii) \quad \text{rank}(Bw, ABw, \ldots, A^{l-1}Bw) = l. \end{aligned} \tag{32}$$

To prove i) it is noted that rank $p_l(A) = p-1$. In fact, since eigenvalues of $p_l(A)$ are $p_l(\lambda_i)$, with $\lambda_i, i = 1, 2, ..., n$ being the eigenvalues of A it follows that matrix $p_l(A)$ has a null eigenvalue with algebraic multiplicity l and unitary geometric multiplicity. Therefore rank $p_l(A) = n - \text{rank}N(p_l(A))$ $=p-1$. Now rank $p_l(A)B <$ rank $p_l(A)$, and therefore i) is satisfied for a p-vector $w \neq 0$.
Let $\hat{w} \neq 0$ be a p-vector solution of i); it is proved that ii) for $w = \hat{w}$ is almost always satisfied. In fact, in opposite case, the minimal polynomial $\hat{p}_l(s)B\hat{w}$, with respect to A, would have a degree $\hat{l} < l$; since moreover $\hat{p}_l(s)$ is a divisor of the characteristic polynomial of A, then by a procedure quite similar to the previous one, it would be

$$\text{rank } \hat{p}_l(A) \geq p. \tag{33}$$

From (33) it would follow that

$$\dim \Re(B) + \dim N(\hat{p}_l(A)) \leq n \tag{34}$$

and therefore

$$\dim\{\Re(B) \cap N(\hat{p}_l(A)\} = 0 \tag{35}$$

for almost any matrix B. Eq. (35) would imply

$$\text{rank } (\hat{p}_l(A)B = p \tag{36}$$

and so $\hat{w}=0$, against the assumption. The Lemma 1 is proved.
From Lemma 1 it follows

Lemma 2. Let the system (1) and $\nu-1$ symmetric eigenvalues of *A* be given with $\nu=m$ ($\nu=p$ respectively); if the remaining *l* eigenvalues of *A*, with $l=n-\nu+1$, have a unitary geometric multiplicity, then for almost any pair (*B*, *C*) there exist a matrix *K* such that $\hat{A}$ has, among its own eigenvalues, the given $\nu-1$ eigenvalues of *A*, along with other $\hat{q}=\min(n,q)-\nu+1$ eigenvalues arbitrarily close to $\hat{q}$ specified symmetric values.

Proof. Let us consider the case $\nu=m$. Let $\Lambda_1=\{\lambda_1,\ \lambda_2,...,\ \lambda_{m-1}\}$ be the set of eigenvalues of *A* which must be retained in $\hat{A}$, and $\Lambda_2=\{\lambda_m,...,\ \lambda_n\}$ the set of remaining eigenvalues. Let:

$$p_2(s)=s^l+a_1 s^{l-1}+\ldots+a_j \tag{37}$$

be the polynomial whose roots hum the set Λ_2.
From Lemma 1 the existence of a vector $c\in\Re(C^T)$ is assured so that minimal polynomial of *c*, with respect to A^T, is $p_2(s)$ for almost any matrix C^T.
Let *D* be now a matrix so that matrix

$$T=\begin{bmatrix} c & A^T c & ... & (A^T)^{l-1}c & D^T \end{bmatrix}^T \tag{38}$$

is not singular. By the linear transformation $z=Tx$, the following system is achieved

$$\dot{z}=\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} z+\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \quad y=\begin{bmatrix} C_1 & C_2 \end{bmatrix} z, \tag{39}$$

where

$$A_{11}=\begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \\ -a_l & -a_{l-1} & . & ... & -a_1 \end{bmatrix} \qquad (40)$$

and A_{22} is an $(m-1)\times(m-1)$ matrix with spectrum Λ_1.
Let $c=C^T w$, with w a non null vector solution of the equation

$$p_2(A^T)C^T w=0\,. \qquad (41)$$

By applying as input to system (39) the control $u=kw^T y$, the closed loop dynamic matrix becomes

$$\hat{A}=\begin{bmatrix} A_{11}+B_1 kw^T C_1 & 0 \\ A_{21}+B_2 kw^T C_1 & A_{22} \end{bmatrix}. \qquad (42)$$

.

Matrix (42) shows that eigenvalues of A_{22} remain unchanged, whereas eigenvalues of $\hat{A}_{11}$ can be altered by varying *k*; for this purpose it is noted that

$$w^T C_1=\begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix} \qquad (44)$$

and then from (40) it follows that the pair $(A_{11},\, w^T C_1)$ is observable.
Moreover, for almost any pair (*B*, *C*), the matrix

$$B_1^T=B^T\begin{bmatrix}c & A^T c & \dots & (A^T)^{l-1}c\end{bmatrix} \qquad (45)$$

is of full rank, i.e. rank $B_1=\min(n,q)-m+1$. On account of Theorem 1 it follows that then exists such a vector *k* that $\hat{q}$ eigenvalues of

$$\hat{A}_{11}=A_{11}+B_1 kw^T C_1 \qquad (46)$$

are arbitrarily close to $\hat{q}$ specified symmetric values. Lemma 2 is then proved in case $\nu = m$; if $\nu = p$, it is sufficient to apply the previous results to matrix $(A + BKC)^T$.

Theorem 3 easily follows from theorem 1 and Lemma 2.

In a first step, a matrix K_1 is found by applying theorem 1, so that $(m-1)$ or $(p-1)$ eigenvalues of $A + BK_1C$ are assigned; if necessary K_1 is slightly modified so that the remaining eigenvalues have unitary geometric multiplicity [8].

In a further step, by applying Lemma 2, a matrix K_2 is determined so that the $(m-1)$ or $(p-1)$ eigenvalues already set, are retained in $A + B(K_1 + K_2)C$ and the other $\hat{q}$ eigenvalues are assigned arbitrarily close to $\hat{q}$ specified symmetric values.

Remark. If $\min(n,q) < n$, fewer eigenvalues of $\min(n,q)$ can be exactly set (by applying (19) as in case *b*, in first and second step or in second only) and degrees of freedom can be used whenever they turn out to be useful for setting adequate limits to non preassigned eigenvalues.

The following example, derived from [5], describes the aforegoing.

Example 2. Let the reachable and observable system (1) be given with:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{47}$$

A matrix *K* is required so that $A + BK_1C$ has eigenvalues arbitrarily close to $\{-1,\ -2,\ -5\}$. By using (19) to assign two eigenvalues $\{-1,\ -2\}$ the result is as follows:

.

$$K_1 = \begin{bmatrix} 0 & 0 \\ 6 & 7 \end{bmatrix}, \; A_1 = A + BK_1C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 7 & 0 \end{bmatrix}. \tag{49}$$

Let $\Lambda_1 = \{-1\}$; then $p_2(s) = s^2 - s - 6$ and Eq. (41) with $A = \Lambda_1$ becomes

$$\begin{bmatrix} -6 & 6 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0 \,. \tag{50}$$

It follows:

$$w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}. \tag{51}$$

By solving (19) for the second step we have

$$K_2 = \begin{bmatrix} -4 & -4 \\ -16 & -16 \end{bmatrix} \tag{52}$$

and then

$$K = K_1 + K_2 = \begin{bmatrix} -4 & -4 \\ -10 & -9 \end{bmatrix} \tag{53}$$

is the desired feedback matrix such that $\hat{A}$ has as its own eigenvalue $\{-1, \ -2, \ -5\}$.

5 Conclusions

In this paper proofs of some fundamental theorems on incomplete pole assignment have been developed.
In such proofs no preliminary transformation of the dynamic system in Jordan form is required.
All the theorems are proved in a unified manner, and the role played by matrix [*F*, *G*] is shown. Moreover, the link between matrix *F* and the reachability matrix with respect to output space is given explicitly.

From the procedure presented an algorithm is derived for incomplete pole assignment. Such an algorithm allows to assign some poles to the other closed loop poles, simultaneously singled out. Degrees of freedom in synthesis are moreover pointed out whenever they turn out to be useful for setting adequate limits to non preassigned poles.

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STABILIZATION BY DIGITAL CONTROLLERS OF MULTIVARIABLE LINEAR SYSTEMS WITH TIME-LAGS

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Abstract. The problem of stabilizing linear systems with time-lags is investigated. Under rather general assumption it is shown that, by means of a digital controller, the system can be always reduced to a system with mere delay connected in cascade with a subsystem whose poles can be arbitrarily assigned.

Keywords. Time lag systems; digital control; pole placement; multivariable control systems; stability.

1 Introduction

In this paper the problem of stabilizing linear systems with time-lags is investigated. Le the system be represented by the discrete model:

$$\begin{aligned} x(kT+T) &= Ax(kT) + Bu(kT-h') \\ y(kT) &= Cx(kT-h''), \end{aligned} \tag{1}$$

where $x \in R^n$ is the state vector, $u \in R^p$ and $y \in R^q$ are the input and output vectors, respectively; $h' \geq 0$ is time-lag in control action, $h'' \geq 0$ is time-lag in output measurement, T is the sampling period.

Models of this type arise quite frequently in technical practice, e.g. sampled-data systems, remote control, control of some industrial process, biological and economic systems, etc.

By increasing time-lags h' and h'', if necessary, through additional delaying devices on input and output of system (1), it is always possible to realize:

$$h' = m'T, \quad h'' = m''T, \tag{2}$$

where m' and m'' are integers. Then the discrete model (1) can be rewritten as:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_{k-m'} \\ y_k &= Cx_{k-m''}, \end{aligned} \tag{3}$$

where, for sake of notational simplicity, the period T is omitted.
The problem to face is one of finding a digital controller allowing the stabilization of discrete system (3).
Assuming that $rankB = p$ and $rankC = q$ system (3) turns out to be equivalent to a discrete linear time invariant system of order $n + pm' + qm''$ without time-lags; therefore, the different techniques of pole assignment (Wonham, 1974; Davison, 1975; Balestrino, Celentano and Sciavicco, 1976) could be used for stabilizing purposes. In this way, however, controllers with high dimensions would result.
Now the question arises of solving the stabilization problem by means of a digital controller with a convenient simple structure.
Hereinafter it is shown that if matrix A is cyclic, pair (A, B) reachable and pair (A, C) observable, the a digital controller can be designed so that the closed loop transfer matrix shows a total time-lag $m = m' + m''$, whereas other poles can be arbitrarily assigned. Of course, this result holds true also for continuous time invariant linear systems with time-lags if the corresponding sampled-data model is taken into account.

2 Digital controller design with $rankC = n$

The controller recommended here is described by the following equations:

$$\begin{aligned} z_{k+1} &= Wz_k + K_{22}z_{k-m} + K_{21}y_{k-m'} \\ u_k &= K_{11}y_k + K_{12}z_{k-m''} + v_k, \end{aligned} \tag{4}$$

where $m = m' + m''$ is the total time-lag, $z \in R^m$ is an internal variable of the controller, $v \in R^p$ is the external input and $W, K_{11}, K_{12}, K_{21}, K_{22}$ are real constant matrices of appropriate dimensions.
The augmented system, consisting of system (3) and controller (4), is described by:

$$\begin{aligned} g_{k+1} &= A' g_k + B' K C g_{k-m} + B'' v_{k-m'} \\ y_k &= C'' g_{k-m''}, \end{aligned} \tag{5}$$

where:

$$g_k^T = \begin{bmatrix} x_k^T & z_k^T \end{bmatrix} \tag{6}$$

$$A' = \begin{bmatrix} A & 0 \\ 0 & W \end{bmatrix}, \quad B' = \begin{bmatrix} B & 0 \\ 0 & I_m \end{bmatrix}, \quad C' = \begin{bmatrix} C & 0 \\ 0 & I_m \end{bmatrix}, \tag{7}$$

with I_m the identity matrix of order m,

$$B'' = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C'' = \begin{bmatrix} C & 0 \end{bmatrix} \tag{8}$$

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}. \tag{9}$$

For system (3) the following result holds.

Theorem 1. Let system (3) be considered and let it be assumed that A is cyclic, pair (A, B) reachable and $rankC = q = n$. Then a controller of type (4) exists such that the compound system (5) has $n'' = 2m + n$ poles arbitrarily close to n'' specified symmetric values and other m poles in the origin of the complex plane.

In order to prove Theorem 1 the following preliminary results are necessary.

Lemma 1. A necessary and sufficient condition for matrix $A' = blockdiag(A, W)$ to be cyclic is that A and W are both cyclic with separate spectra.

Proof. Let T_A and T_w be two nonsingular matrices such that $T_A^{-1} A T_A = J_A$ and $T_W^{-1} W T_W = J_W$ are in Jordan's form.
Then $\hat{T}^{-1} A' \hat{T}$, with $\hat{T} = blockdiag(T_A, T_W)$, is a matrix in Jordan's form. Demonstration proceeds from the foregoing and from the fact that a matrix M is cyclic if and only if its Jordan's form J_M has just one Jordan's block in correspondence with the same eigenvalue.

Lemma 2. If pair (A, B) is reachable, then also pair (A', B'), see (7), is reachable.

The proof is trivial and therefore omitted.

Proof of Theorem 1. Let matrix W be chosen cyclic with spectrum separate from one of A. Thus, A' is cyclic due to Lemma 1. Moreover, pair (A', B') is reachable due to Lemma 2; therefore a vector $b' = B' f$ exists with $f \in R^{p+m}$ so that pair (A', b') is reachable (Davison, Wang, 1973).
Let

$$p_{A'}(\lambda) = \lambda^{n'} + a_1' \lambda^{n'-1} + ... + a_{n'}' \tag{10}$$

be the characteristic polynomial of A', with $n' = n + m$.
Let the nonsingular matrix

$$T_c = \psi \begin{bmatrix} a_{n'-1}' & a_{n'-2}' & ... & a_1' & 1 \\ a_{n'-2}' & a_{n'-3}' & ... & 1 & 0 \\ . & . & ... & . & . \\ a_1' & 1 & ... & 0 & 0 \\ 1 & 0 & ... & 0 & 0 \end{bmatrix} \tag{11}$$

be considered, where

$$\psi = \left[b' \quad A'b' \quad ... \quad A'^{n'-1} b' \right]; \tag{12}$$

moreover

$$K = fk^T T_c^{-1} C'^{-1} \tag{13}$$

be chosen, where k is a n'-vector.
Now transfer matrix of system (5), given by

$$D(\zeta) = C''(\zeta^{m+1} I - A'\zeta^m - B'KC')^{-1} B'' \tag{14}$$

can be rewritten as follows

$$D(\zeta) = C'' T_c (\zeta^{m+1} I - \hat{A}\zeta^m - \hat{b}k^T)^{-1} T_c^{-1} B'' \ , \tag{15}$$

where:

$$\hat{A} = T_c^{-1} A' T_c = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \\ -a'_{n'} & -a'_{n'-1} & . & ... & -a'_1 \end{bmatrix} \tag{16}$$

$$\hat{b} = T_c^{-1} B' f = \begin{bmatrix} 0 & 0 & ... & 0 & 1 \end{bmatrix}^T \ . \tag{17}$$

From (15), (16) and (17), after standard manipulations, it follows that poles of $D(\zeta)$ are the zeros of polynomial

$$d(\lambda) = \lambda^m p(\lambda), \tag{18}$$

where

$$p(\lambda) = \lambda^m p_{A'}(\lambda) - \begin{bmatrix} 1 & \lambda & \dots & \lambda^{n'-1} \end{bmatrix} k =$$
$$= \lambda^{n''} + a_1^{'} \lambda^{n''-1} + \dots + a_m^{'} \lambda^{n'} + (a_{m+1}^{'} - k_{n'}) \lambda^{n'-1} + \qquad (19)$$
$$+ \dots + (a_{n'}^{'} - k_{m+1}) \lambda^m - k_m \lambda^{m-1} + \dots - k_1,$$

k_j being the $j-th$ component of feedback vector k. Then from (19) it follows that the first m coefficient of $p(\lambda)$ coincide with the first m coefficient of $p_{A'}(\lambda)$ and cannot therefore be modified by feedback vector k, whereas the remaining ones can be arbitrarily modified by a suitable choice of vector k.

Let

$$p_A(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_n \qquad (20)$$

and

$$p_W(\lambda) = \lambda^m + w_1 \lambda^{m-1} + \dots + w_n$$
(21)

be characteristic polynomials of A and W, respectively.

Since $A' = blockdiag(A, W)$

$$p_{A'}(\lambda) = p_A(\lambda) p_W(\lambda) ; \qquad (22)$$

then it follows (Balestrino, Celentano, Sciavicco, 1976)

$$\begin{bmatrix} a_1^{'} - a_1 \\ a_2^{'} - a_2 \\ . \\ a_n^{'} - a_n \\ a_{n+1}^{'} \\ . \\ a_{n'}^{'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & . & 0 \\ a_1 & 1 & . & 0 \\ . & . & . & . \\ a_n & a_{n-1} & . & a_{.} \\ 0 & a_n & . & a_{.} \\ . & . & . & . \\ 0 & 0 & . & a_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ . \\ w_m \end{bmatrix}. \qquad (23)$$

Now let

$\hat{p}(\lambda) = \lambda^{n"} + \hat{a}_1 \lambda^{n"-1} + ... + \hat{a}_{n"}$
(24)

be a real coefficient polynomial whose roots are the desired poles with $n" = 2m + n.$
In order to have $p(\lambda) = \hat{p}(\lambda)$, from (19) it follows that:

$$a_i^{'} = \hat{a}_i, \quad i = 1, 2, ..., m; \tag{25}$$

moreover

$$k_j = \begin{cases} -\hat{a}_{n"+1-j}, & j = 1, 2, ..., m \\ -\hat{a}_{n"+1-j} + a^{'}_{n"+1-j}, & j = m+1, 2, ..., n'. \end{cases} \tag{26}$$

From (23) and (25) it follows that polynomial (24) is obtained if

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ . \\ w_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & . & 0 & 0 \\ a_1 & 1 & 0 & . & 0 & 0 \\ a_2 & a_1 & 1 & . & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & a_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{a}_1 - a_1 \\ \hat{a}_2 - a_2 \\ \hat{a}_3 - a_3 \\ . \\ \hat{a}_m - . \end{bmatrix}. \tag{27}$$

After polynomial $p_W(\lambda)$ is known, matrix W must be determined.
Matrix W can be chosen in companion form; then the matrix W is cyclic and for almost any $\hat{p}(\lambda)$ it's spectrum is separate from one of A. If for a specified $\hat{p}(\lambda)$ the spectrum of W is not separate from one of A, poles to assign can be chosen as $\lambda_i + \Delta\lambda_i$, with $\Delta\lambda_i \to 0$, $i = 1, ..., n"$, so that the spectra are separate.
The proof of Theorem 1 is complete.

3 Digital controller design with $rankC < n$

In this case, too, the problem of pole assignment of system (3) can be solved if an asymptotic observer of vector $x_{k-m''}$ can be used.
In order to prove this statement, the following preliminary results must be considered.

Lemma 3. Let pair (A,C) be observable, with $A \in R^{n\times n}$, $C \in R^{q\times n}$ and $rankC = q < n$. Let $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_{n-q}\}$ be a symmetric set of complex numbers.
Three matrices then exist, S, P, Q with dimension $(n-q)\times(n-q)$, $n\times q$, $(n-q)\times(n-q)$, respectively, so that

$$Q(A-PC) = SQ \ , \tag{28}$$

with Λ coincident with spectrum of S and

$$rank\begin{bmatrix} C \\ Q \end{bmatrix} = n \ . \tag{29}$$

For the relative proof see Wonham (1974, pp. 60-62).

Lemma 4. Let the dynamic linear time invariant system

$$s_{k+1} = Ss_k + QPy_k + QBu_{k-m} \tag{30}$$

be considered, where S, P, Q are given by Lemma 3 with the spectrum of S inside the unit circle and u_k, y_k are, respectively, the input and output vectors of system (3). Then

$$\begin{bmatrix} C \\ Q \end{bmatrix}^{1} \begin{bmatrix} y_k \\ s_k \end{bmatrix} \tag{31}$$

represents an asymptotic estimate of vector $x_{k-m''}$ of system (3).

Proof. Let

$$e_k = s_k - Qx_{-m''} \,. \tag{32}$$

From (3), (30), by means of (28)

$$e_{k+1} = Se_k \tag{33}$$

is obtained.
Since the spectrum of S has been assumed to be inside the unit circle, there results

$$\lim_{k\to\infty} e_k = 0 \,. \tag{34}$$

From (3), (29), (32), (34) it follows that

$$\lim_{k\to\infty} \begin{bmatrix} C \\ Q \end{bmatrix}^{-1} \begin{bmatrix} y_k \\ s_k \end{bmatrix} = x_{k-m''} \,. \tag{35}$$

Remark 1. If, in Lemma 4, S is chosen nilpotent then the asymptotic observer is of dead-beat type.

The digital controller can now be synthetized. It can be specified by the following equations:

$$\begin{aligned} z_{k+1} &= Wz_k + K_{22} z_{k-m} + K_{21} \begin{bmatrix} y_{k-m'} \\ s_{k-m'} \end{bmatrix} \\ s_{k+1} &= Ss_k + QPy_k + QBu_{k-m} \\ u_k &= K_{11} \begin{bmatrix} y_k \\ s_k \end{bmatrix} + K_{12} z_{k-m''} + v_k \,, \end{aligned} \tag{36}$$

where matrices $W, K_{11}, K_{12}, K_{21}, K_{22}$ have the same dimensions as considered in the previous section; moreover, P, Q, S are the matrices as identified in Lemma 4 and v_k is the external input.
By setting

$$e_k = s_k - Qx_{k-m''} \tag{37}$$

the augmented system consisting of system (3) and controller is described by:

$$\begin{aligned} g_{k+1} &= A' g_k + B' K C' g_{k-m} + E e_{k-m'} + B'' v_{k-m'} \\ e_{k+1} &= S e_k \\ y_k &= C'' g_{k-m''} , \end{aligned} \tag{38}$$

where g_k is a vector defined by (6), A', B', C', B'', C'', K are matrices defined by (7), (8), (9), where, however, in the last matrix of (7) matrix C must be replaced by $\begin{bmatrix} C^T & Q^T \end{bmatrix}^T$; moreover

$$E = \begin{bmatrix} BK_{11} \begin{bmatrix} 0 \\ I_{n-q} \end{bmatrix} \\ K_{21} \begin{bmatrix} 0 \\ I_{n-q} \end{bmatrix} \end{bmatrix} . \tag{39}$$

For system (38) the following result holds, like that of Theorem 1.

Theorem 2. Let system (3) be considered and let it be assumed that A is cyclic, pair (A,B) reachable, pair (A,C) observable and $rank C = q < n.$ Then a controller of type (36) exists so that the compound system (38) has $n'' + (n-q)$ poles arbitrarily close to $n'' + (n-q)$ specified symmetric values and other m poles in the origin of complex plane.

Proof. It is immediately shown that poles of system (36) coincide with eigenvalues of S and with poles of transfer function

$$D(\zeta) = C''(\zeta^{m+1} I - A'\zeta^{m} - B'KC')^{-1} B'' \ . \qquad (40)$$

Demonstration follows from Lemma 3 and Theorem 1.

Remark. A dual of Theorem 2 can be derived by means of a dual observer (Luenberger, 1966). Thus, s_k , is a $(n-p)-$ vector, with $p = rankB.$ Practically, the better of two results can be used to implement the controller.

4 Conclusions

In this paper the problem of stabilizing linear systems with time-lags has been investigated.
Under rather general assumptions, it has been shown that it is always possible, by means of a digital controller, to reduce the system to a system with mere delay cascade connected with a subsystem whose poles can arbitrarily assigned.

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CONTROLLORI DINAMICI INTERAGENTI

1 Introduzione e risultati preliminari

Si consideri il sistema lineare, stazionario, raggiungibile ed osservabile descritto dalle equazioni:

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{1}$$

in cui $x \in R^n$ è lo stato, $u \in R^r$ è l'ingresso, $y \in R^m$ è l'uscita ed A, B, C sono matrici reali di dimensioni opportune con $rango B = r$ e $rango C = m.$ Retroazionando tale sistema mediante il controllore (detto anche compensatore) dinamico descritto dalle equazioni:

$$\dot{w} = Ww - Dy, \quad u = -Ky + Hw + v\,, \tag{2}$$

in cui $w \in R^\nu$ è lo stato, $v \in R^r$ è il nuovo ingresso e W, D, K, H sono matrici reali di dimensioni opportune, il sistema complessivo risulta:

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A - BKC & BH \\ -DC & W \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v, \quad y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}. \tag{3}$$

Questo capitolo tratta il problema dell'assegnamento dei poli del sistema complessivo senza suddividerlo in due sottosistemi; più precisamente, esso studia l'assegnabilità degli autovalori della matrice

$$A_c = \begin{bmatrix} A - BKC & BH \\ -DC & W \end{bmatrix} \tag{4}$$

mediante opportuna scelta delle matrici $K, H, D, W.$

A tale scopo si premette il seguente lemma.

Lemma 1. Sia

$$A=\begin{bmatrix} 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & 1 \\ -a_n & -a_{n-1} & . & . & -a_1 \end{bmatrix} \qquad (5)$$

una matrice reale $n\times n$ in forma compagna e C una matrice reale $m\times n$ qualsiasi. Posto

$$a=\begin{bmatrix} a_n & a_{n-1} & \dots & a_1 & 1 \end{bmatrix}^T \qquad (6)$$

e detto ν un intero non negativo si ha

$$rango\left(\begin{bmatrix} S_{\nu+1}(C^T) & S_\nu(a) \end{bmatrix}\right)=rango\left(\begin{bmatrix} C^T & A^TC^T & \dots & (A^T)^\nu C^T \end{bmatrix}\right)+\nu\ , \qquad (7)$$

ove $S_i(M)$, $M\in R^{n\times m}$, denota la matrice di dimensioni $(n+i-1)\times(m\cdot i)$

$$S_i(M)=\underbrace{\begin{bmatrix} M & 00...0 & \cdots & 00...0 \\ 00...0 & M & \cdots & 00...0 \\ 00...0 & 00...0 & & \cdots \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & \cdots & M \\ 00...0 & 00...0 & \cdots & \end{bmatrix}}_{m\cdot i}\ \Big\}\, n+i-1 \quad .$$

Dimostrazione. Viene omessa per brevità.

2 Assegnamento mediante controllore di ordine $\min\{\nu_r, \nu_o\}$

Il Lemma 1 consente di stabilire il seguente teorema.

Teorema 1. Relativamente al sistema (1) si può progettare un controllore dinamico del tipo (2) di ordine ν tale che $\eta = \max\{r_\nu, o_\nu\} + \nu$ poli del sistema complessivo (3), di ordine $n+\nu$, siano arbitrariamente vicini ad η specificati valori simmetrici, dove

$$r_\nu = rango\left(\begin{bmatrix} B & AB & \dots & A^\nu B \end{bmatrix}\right),\quad o_\nu = rango\left(\begin{bmatrix} C^T & A^T C^T & \dots & (A^T)^\nu C^T \end{bmatrix}\right). \tag{8}$$

Dimostrazione. Viene omessa per brevità.

Il Teorema 1 contiene, come caso particolare, il seguente risultato fondamentale.

Teorema 2. Mediante un controllore dinamico di ordine $\nu = \min\{\nu_r, \nu_o\}$, dove $\nu_r + 1$ (risp. $\nu_o + 1$) è l'indice di raggiungibilità (risp. osservabilità) del sistema (1), è possibile assegnare arbitrariamente tutti i poli del sistema complessivo (3).

Dimostrazione. E' immediata.

Nel caso in cui $\nu < \min\{\nu_r, \nu_o\}$ dal Teorema 1 segue che $n - \max\{r_\nu, o_\nu\}$ poli del sistema complessivo non possono essere assegnati ad arbitrio. Se la posizione di questi nel piano complesso non dovesse risultare accettabile, o si aumenta l'ordine ν del controllore oppure si può ricorrere al seguente teorema, che è una generalizzazione del Teorema 1.

Teorema 3. Mediante un controllore dinamico di ordine $\nu < \min\{\nu_r, \nu_o\}$ è possibile assegnare $\eta < \max\{r_\nu, o_\nu\} + \nu$ poli del sistema (3) arbitrariamente vicini ad η specificati valori simmetrici; inoltre i rimanenti $n + \nu - \eta$ poli sono vincolati dall'equazione

$$r(\lambda) = q_o(\lambda) + h_1 q_1(\lambda) + \dots + h_l q_l(\lambda) = 0, \tag{9}$$

dove:

$$l = \begin{cases} m(\nu+1) + \nu - \eta, & \text{se } r_\nu \le \text{o}_\nu \\ r(\nu+1) + \nu - \eta, & \text{se } r_\nu > \text{o}_\nu \end{cases}, \tag{10}$$

$q_o(\lambda), q_1(\lambda), ..., q_l(\lambda)$ sono degli opportuni polinomi di grado al più $n+\nu-\eta$ ed $h_1, h_2, ..., h_l$ sono scalari che possono essere assegnati ad arbitrio lasciando inalterati i prefissati η poli.

Dimostrazione. Viene omessa per brevità.

Un algoritmo di progetto di un compensatore di ordine ν per l'assegnamento di $\eta \le \max\{r_\nu, o_\nu\}+\nu$ poli (cfr. Teoremi 1, 2, 3) è il seguente.

Algoritmo 1.

Passo 1. Se $r_\nu > o_\nu$ si sostituisca la terna (A, B, C) con la duale (A^T, C^T, B^T) altrimenti si va al passo successivo.

Passo 2. Si scelgano ad arbitrio una matrice $K_0 \in R^{r\times m}$ ed un vettore $f \in R^r$ tali che la coppia $(A+BK_0C, Bf)$ sia raggiungibile. Se A è ciclica si può scegliere $K_0 = 0$.

Passo 3. Si calcolino l' $(n+1)$ -vettore

$$a = \begin{bmatrix} a_n & a_{n-1} & ... & a_1 & 1 \end{bmatrix}^T \tag{11}$$

e la matrice $\hat{C} \in R^{m\times n}$ mediante la formula

$$\hat{C} = \begin{bmatrix} Bf & (A+BK_0C)Bf & ... & (A+BK_0C)^{n-1}Bf \end{bmatrix} \begin{bmatrix} a_{n-1} & a_{n-2} & . & a_1 & 1 \\ a_{n-2} & a_{n-3} & . & 1 & 0 \\ . & . & . & . & . \\ a_1 & 1 & . & 0 & 0 \\ 1 & 0 & . & 0 & 0 \end{bmatrix}. \tag{12}$$

Passo 4. Si costruiscano le matrici F, G ed L come segue:

$$F = S_{\nu+1}(\hat{C}^T) \in R^{(n+\nu)\times m(\nu+1)}, \quad G = S_\nu(a) \in R^{(n+\nu)\times \nu}, \quad L = S_{n+\nu-\eta}(d) \in R^{(n+\nu)\times(n+\nu-\eta)}, \tag{13}$$

ove $d = \begin{bmatrix} d_\eta & d_{\eta-1} & ... & d_1 & 1 \end{bmatrix}^T$ è il vettore dei coefficienti del polinomio $d(\lambda) = \lambda^\eta + d_1\lambda^{\eta-1} + ... + d_{\eta-1}\lambda + d_\eta$ che ha per radici gli η poli desiderati.

Passo 5. Si risolva il sistema

$$\begin{bmatrix} F & G & L \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ . \\ q_{\nu+1} \\ \pi_0 \\ -\rho \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ d_\eta \\ . \\ d_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ . \\ a_n \\ . \\ a_1 \end{bmatrix}, \quad q_i \in R^m, \ \pi_0 \in R^\nu, \rho \in R^{n+\nu-\eta} \tag{14}$$

lasciando liberi $l = n + \nu - \eta$ elementi $h_1, h_2, ..., h_l$ di q_i, π_0, ρ, ottenendo così per ρ un'espressione del tipo

$$\rho = \begin{bmatrix} \rho_{n+\nu-\eta} & \rho_{n+\nu-\eta-1} & ... & \rho_1 \end{bmatrix}^T = \chi_0 + h_1\chi_1 + ... + h_l\chi_l \,. \tag{15}$$

Passo 6. Applicando la tecnica del luogo delle radici o il criterio di Routh al polinomio

$$r(\lambda) = \lambda^{n+\nu-\eta} + \rho_1\lambda^{n+\nu-\eta-1} + ... + \rho_{n+\nu-\eta} = r_0(\lambda) + h_1 r_1(\lambda) + ... + h_l r_l(\lambda), \tag{16}$$

si fissino i valori $h_1, h_2, ..., h_l$ in modo che i suoi zeri, che sono i restanti poli del sistema complessivo (3), abbiano una configurazione soddisfacente.

Passo 7. Si determinino k_0 e D risolvendo l'equazione

$$[k_0 \quad D] = \begin{bmatrix} w_\nu & w_{\nu-1} & . & w_1 & 1 \\ w_{\nu-1} & w_{\nu-2} & . & 1 & 0 \\ . & . & . & . & . \\ w_1 & 1 & . & 0 & 0 \\ 1 & 0 & . & 0 & 0 \end{bmatrix} = [q_1 \quad q_2 \quad \dots \quad q_{\nu+1}], \tag{17}$$

dove $[w_\nu \quad w_{\nu-1} \quad \dots \quad w_1] = \pi_0^T$.

Passo 8. Il desiderato controllore (2), se $r_\nu \le o_\nu$, è definito dalle matrici:

$$W = \begin{bmatrix} 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & 1 \\ -w_\nu & -w_{\nu-1} & . & . & -w_1 \end{bmatrix}, \quad D = D, \quad K = K_0 + fk_0^T, \quad H = f[1 \quad 0 \quad \dots \quad 0]; \tag{18}$$

se invece $r_\nu > o_\nu$ il controllore è definito dalle corrispondenti matrici trasposte.

3 Assegnamento mediante controllore di ordine ridotto

I risultati stabiliti in precedenza possono essere migliorati se $\min\{m, r\} > 1$. Infatti vale il seguente teorema.

Teorema 4. Per quasi tutti i sistemi (1) si può progettare un controllore di ordine ν che assegna al sistema complessivo (3), di ordine $n+\nu$, $\eta \le \min\{m + r - 1 + \nu \max\{m, r\}, n\} + \nu$ poli arbitrariamente vicini ad η specificati valori simmetrici; inoltre i rimanenti $n+\nu-\eta$ poli sono vincolati dall'equazione

$$r(\lambda) = q_o(\lambda) + h_1 q_1(\lambda) + \dots + h_l q_l(\lambda) = 0, \tag{19}$$

dove:

$$l == \begin{cases} m(\nu+1)+\nu+r-1-\eta, \text{ se } r \le m \\ r(\nu+1)+\nu+m-1-\eta, \text{ se } r > m \end{cases}, \tag{20}$$

$q_o(\lambda), q_1(\lambda), ..., q_l(\lambda)$ sono degli opportuni polinomi di grado al più $n+\nu-\eta$ ed $h_1, h_2, ..., h_l$ sono scalari che possono essere assegnati ad arbitrio lasciando inalterati i prefissati η poli.

Dimostrazione. Mediante il Teorema 2.1 del Capitolo 3 si determinino una matrice $K_0 \in R^{r\times n}$ tale che la matrice $A_0 = A + BK_0C$ sia ciclica con $r-1$ autovalori arbitrariamente vicini ad $r-1$ specificati valori simmetrici ed il polinomio

$$\hat{p}(\lambda) = \lambda^{m+1-r} + \alpha_1\lambda^{n-r} + ... + \alpha_{n+1-r} \tag{21}$$

degli autovalori residui.
Si determini quindi un vettore non nullo $f \in R^r$ tale che

$$\hat{p}(A_0)Bf = 0. \tag{22}$$

Allora, ponendo:

$$K = fk^T, \quad H = fh^T, \quad h, k \in R^m, \tag{23}$$

dopo un opportuno cambiamento di variabile (cfr. Lemma 4.1 del Capitolo 3), la matrice dinamica (4) diventa

$$\begin{bmatrix} A_{11} - b_1k^TC_1 & A_{12} - b_1k^TC_2 & b_1h^T \\ 0 & A_{22} & 0 \\ -DC_1 & -DC_2 & W \end{bmatrix}. \tag{24}$$

Gli autovalori di tale matrice sono quelli di A_{22}, che sono stati assegnati ad arbitrio, e quelli della matrice

$$\begin{bmatrix} A_{11} - b_1 k^T C_1 & b_1 h^T \\ -DC_1 & W \end{bmatrix}. \tag{25}$$

Per il Teorema 3 ed il Lemma 4.1 del Cap. 3, per quasi tutti i sistemi (1), mediante opportuna scelta di k, h, D e W

$$\hat{\eta} \le \nu + \min\{(\nu+1)m, n+1-r\} \tag{26}$$

autovalori della matrice (25) possono essere assegnati arbitrariamente vicini ad $\hat{\eta}$ specificati valori simmetrici con i rimanenti vincolati dall'equazione (21).
Il Teorema resta così dimostrato per il caso $r \le m$; per il caso $r > m$ la dimostrazione segue in maniera analoga considerando la trasposta della matrice dinamica (4).

L'Algoritmo 1 e la dimostrazione del Teorema 3 consentono di dare il seguente algoritmo di progetto.

Algoritmo 2.

Passo 1. Se $r > m$ si sostituisca la terna (A, B, C) con la sua duale (A^T, C^T, B^T) altrimenti si vada al passo successivo.

Passo 2. Mediante l'Algoritmo 3.1 del Capitolo 3 si calcolino una matrice $K_0 \in R^{r \times n}$ tale che $A + BK_0C$ sia ciclica con $r-1$ auto valori arbitrariamente vicini ad $r-1$ specificati valori simmetrici ed il polinomio

$$\hat{p}(\lambda) = \lambda^{m+1-r} + \alpha_1 \lambda^{n-r} + ... + \alpha_{n+1-r} \tag{27}$$

degli auto valori residui.

Passo 3. Si calcolino un vettore non nullo $f \in R^r$ tale che

$$\hat{p}(A_0)Bf = 0, \tag{28}$$

l' $(n-r)$-vettore

$$a=\left[\alpha_{n+1-r} \quad \alpha_{n-r} \quad \dots \quad \alpha_1 \quad 1\right]^T \tag{29}$$

e la matrice $\hat{C} \in R^{n\times(n+1-r)}$ mediante la formula

$$\hat{C}=\left[Bf \quad (A+BK_0C)Bf \quad \dots \quad (A+BK_0C)^{n-r}Bf\right]\begin{bmatrix} a_{n-r} & a_{n-r-1} & . & a_1 & 1 \\ a_{n-r-1} & a_{n-r-2} & . & 1 & 0 \\ . & . & . & . & . \\ a_1 & 1 & . & 0 & 0 \\ 1 & 0 & . & 0 & 0 \end{bmatrix}. \tag{30}$$

Passo 4. Si eseguano i passi $4\div 8$ dell'Algoritmo 1 sostituendo ad n $n+1-r$ e ad η $\eta+1-r$.

Bibliografia

Il Lemma 1 è dovuto a Balestrino e Celentano [3]. I Teoremi 1 e 2 sono dovuti, rispettivamente, ad Ahmari e Vacroux [2] e Brash e Pearson [2]; le dimostrazioni qui riportate insieme con i Teoremi 2 e 3 si possono trovare in Balestrino e Celentano [3].

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